Development of the Identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

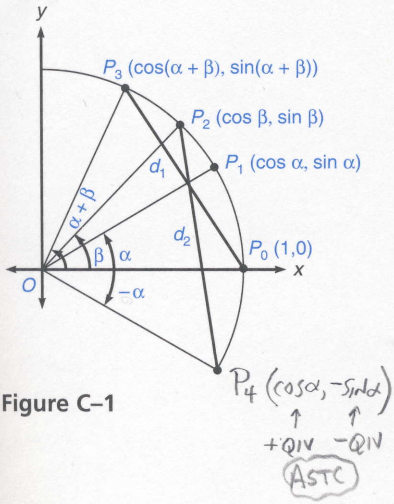

Figure C–1

A proof that this identity is true is beyond the scope of this text, but an argument for its correctness can be obtained in the following way.

Let α and β be two angles in standard position (see figure C.1). Let P_1 be the point where the terminal side of α intersects the unit circle, and P_2 be the point where angle β intersects the unit circle. Let P_3 be the point where the angle $\alpha + \beta$ (the sum of the angles α and β) intersects the circle. Let P_0 be the point (1,0). Finally, let P_4 be the point where the terminal side of angle $-\alpha$ intersects the unit circle.

On the unit circle the x- and y-coordinates of a point are the cosine and sine values for the appropriate angle. Thus, the point P_1 has coordinates $(\cos\alpha, \sin\alpha)$. The coordinates for the other points are shown in the figure.

Angle $\alpha + \beta$, or angle P_0OP_3 in standard position, has the same measure as angle P_4OP_2. It is a geometric property that central angles of a circle having equal measure will have chords of equal length. Thus, the chords P_3P_0 and P_2P_4 have the same length. The length of a line segment with end points (x_1,y_1) and (x_2,y_2) is given by the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We apply this to the chords mentioned above.
Let d_1 = length of P_3P_0, and d_2 = length of P_2P_4.

$$d_1 = d_2$$

$$\sqrt{(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2}$$
$$= \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - (-\sin\alpha))^2}$$

We now square both sides.

$$[\cos(\alpha + \beta) - 1]^2 + [\sin(\alpha + \beta) - 0]^2 = (\cos\beta - \cos\alpha)^2 + [\sin\beta - (-\sin\alpha)]^2$$

Performing the indicated operations we obtain

$$\cos^2(\alpha + \beta) - 2\cos(\alpha + \beta) + 1 + \sin^2(\alpha + \beta)$$
$$= \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta + \sin^2\alpha$$

Then

$$[\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta)] - 2\cos(\alpha + \beta) + 1$$
$$= (\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) + 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta$$

Using the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$1 - 2\cos(\alpha + \beta) + 1 = 1 + 1 + 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta$$
$$-2\cos(\alpha + \beta) = 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \text{Divide each member by } -2$$

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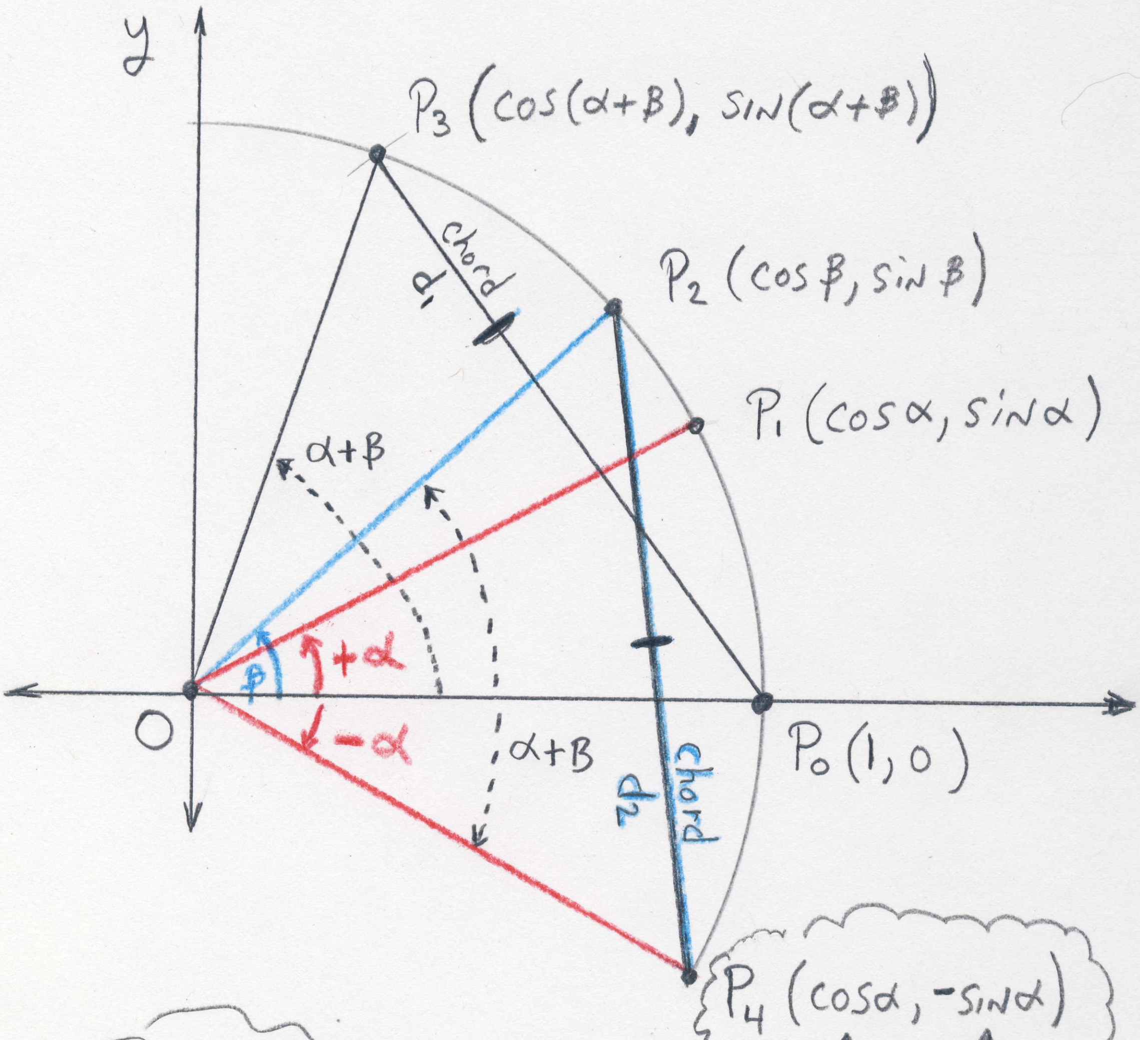
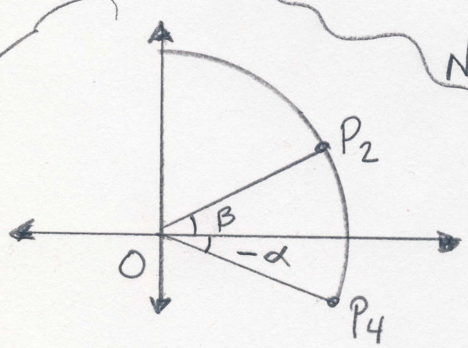
$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$



NOTE:



$m\angle P_4OP_2 = m\angle\alpha + \angle\beta$